Singular solutions to fully nonlinear elliptic equations (Viscosity Solutions of Differential Equations and Related Topics)

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Singular solutions to fully nonlinear elliptic equations

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We study a class of fully nonlinear second-order elliptic equations of the form

$$F(D^2u)=0 \tag{1}$$

defined in a domain of $\mathbf{R}^n$. Here D^2u denotes the Hessian of the function u. We assume that F is a Lipschitz function defined on an open set $D\subset S^2(\mathbf{R}^n)$ of the space of $n\times n$ symmetric matrices satisfying the uniform ellipticity condition, i.e. there exists a constant $C\geq 1$ (called an *ellipticity constant*) such that

$$C^{-1}||N||\leq F(M+N)-F(M)\leq C||N|| \tag{2}$$

for any non-negative definite symmetric matrix N; if $F\in C^1(D)$ then this condition is equivalent to

$$\frac{1}{C'}|\xi|^2\leq F_{u_{ij}}\xi_i\xi_j\leq C'|\xi|^2\ ,\forall\xi\in\mathbf{R}^n\ . \tag{2'}$$

Here, u_{ij} denotes the partial derivative $\partial^2u/\partial x_i\partial x_j$. A function u is called a *classical* solution of (1) if $u\in C^2(\Omega)$ and u satisfies (1). Actually, any classical solution of (1) is a smooth $(C^{\alpha+3})$ solution, provided that F is a smooth (C^{α}) function of its arguments.

For a matrix $S\in S^2(\mathbf{R}^n)$ we denote by $\lambda(S)=\{\lambda_i:\lambda_1\leq\ldots\leq\lambda_n\}\in\mathbf{R}^n$ the set of eigenvalues of the matrix S. Equation (1) is called a Hessian equation ([T1],[T2] cf. [CNS]) if the function $F(S)$ depends only on the eigenvalues $\lambda(S)$ of the matrix S, i.e., if

$$F(S)=f(\lambda(S)),$$

for some function f defined on $\mathbf{R}^n$ and invariant under the permutation of the coordinates.

In other words the equation (1) called Hessian if it is invariant under the action of the group $O(n)$ on $\mathbf{R}^n$: for any $O\in O(n)$

$$F({}^tO\cdot S\cdot O)=F(S). \tag{3}$$

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If we assume that the function $F(S)$ is defined for any symmetric matrix S, i.e., $D = S^2(\mathbf{R}^n)$ the Hessian invariance relation (3) implies the following:

(a) F is a smooth (real-analytic) function of its arguments if and only if f is a smooth (real-analytic) function.

(b) Inequalities (2) are equivalent to the inequalities

$$\frac{\mu}{C_0} \leq f(\lambda_i + \mu) - f(\lambda_i) \leq C_0\mu, \ \forall \mu \geq 0,$$

$\forall i = 1, ..., n$, for some positive constant C_0.

(c) F is a concave function if and only if f is concave, [CNS].

Well known examples of the Hessian equations are Laplace, Monge-Ampère, Bellman and Special Lagrangian equations.

Consider the Dirichlet problem

$$\begin{cases} F(D^2u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \ , \end{cases} \tag{4}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

We are interested in the problem of existence and regularity of solutions to Dirichlet problem (4) for Hessian equations. Dirichlet problem (4) has always a unique viscosity (weak) solutions for fully nonlinear elliptic equations (not necessarily Hessian equations). The viscosity solutions satisfy the equation (1) in a weak sense, and the best known interior regularity ([C],[CC]) for them is $C^{1+\epsilon}$ for some $\epsilon > 0$. For more details see [CC], [CIL]. Until recently it remained unclear whether non-smooth viscosity solutions exist. In [NV1] we proved the existence of viscosity solutions to the fully nonlinear elliptic equations which are not classical in dimension 12. Moreover, we proved in [NV2], that in 24-dimensional space the optimal interior regularity of viscosity solutions of fully nonlinear elliptic equations is no more than $C^{2-\delta}$. Both papers [NV1,NV2] use the function

$$w = \frac{Re(q_1 q_2 q_3)}{|x|},$$

where $q_i \in \mathbf{H}$, $i = 1, 2, 3$, are Hamiltonian quaternions, $x \in \mathbf{H}^3 = \mathbf{R}^{12}$ which is a viscosity solution in $\mathbf{R}^{12}$ of a uniformly elliptic equation (1) with a smooth F.

Our main result shows that the same function w is a solution to a Hessian equation. Moreover the following theorem holds

Theorem 1.1. **(N.Nadirashvili, S.Vladuts)** *For any δ, $0 \leq \delta < 1$ the function*

$$w/|x|^\delta$$

is a viscosity solution to a uniformly elliptic Hessian equation (1) *in a unit ball* $B \subset \mathbf{R}^{12}$.

Theorem 1.1 shows that the second derivatives of viscosity solutions of Hessian equations (1) can blow up in an interior point of the domain and that the optimal interior regularity of the viscosity solutions of Hessian equations is no more than $C^{1+\varepsilon}$, thus showing the *optimality* of the result by Caffarelli and Trudinger [C,CC, T3] on the interior $C^{1,\alpha}$-regularity of viscosity solutions of fully nonlinear equations. Our construction provides a Lipschitz functional F in Theorem 1.1. Using a more complicated argument one can make F smooth; we will return to this question elsewhere. However, if we drop the invariance condition (3) we get

Corollary 1.1. *For any* δ, $0 \leq \delta < 1$ *the function*

$$w/|x|^{\delta}$$

is a viscosity solution to a uniformly elliptic (not necessarily Hessian) equation (1) *in a unit ball* $B \subset \mathbf{R}^{12}$ *where* F *is a* (C^{∞}) *smooth functional.*

Ball B in Theorem 1.1 can not be substituted by the whole space $\mathbf{R}^{12}$. In fact, for any $0 < \alpha < 2$ there are no homogeneous order α solutions to the fully nonlinear elliptic equation (1) defined in $\mathbf{R}^n \setminus \{0\}$, [NY]; the essence of the difference with the local problem is that in the case of homogeneous solution defined in $\mathbf{R}^n \setminus \{0\}$ one deals simultaneously with two singularities of the solution: one at the origin and another at the infinity. In the local problem the structure of singularities of solutions is quite different, even in dimension 2, the function $u = |x|^{\alpha}$, $0 < \alpha < 1$, $x \in B^o$, where B^o is a punctured ball in $\mathbf{R}^n$, $n \geq 2$, $B^o = \{x \in \mathbf{R}^n, 0 < |x| < 1\}$, is a solution to the uniformly elliptic Hessian equation in B^o (notice that u is *is not* a viscosity solution of any elliptic equation on the whole disk B).

Due to Krylov-Evans regularity theory for elliptic equations (1) with convex F all viscosity solutions are smooth. For the Special Lagrangian equation it follows from the main result of [JX] that there is no nontrivial homogeneous order 2 solution. Nonexistence for the Special Lagrangian equation of homogeneous solutions of order $\alpha \neq 2$ follows from [NY].

We study also the possible singularity of solutions of Hessian equations defined in a neighborhood of a point. We prove the following general result:

Theorem 1.2. **(N.Nadirashvili, S.Vladuts)** *Let* u *be a viscosity solution of a uniformly elliptic Hessian equation in a punctured ball* $B^o \subset \mathbf{R}^n$. *Assume that* $u \in C^0(B)$. *Then* $u = v + l + o(|x|^{1+\epsilon})$, *where* v *is a monotone function of the radius,* $v(x) = v(|x|)$, $v \in C^{\epsilon}(B)$, *where* $\epsilon > 0$ *depends on the ellipticity constant of the equation, and* l *is a linear function.*

As an immediate consequence of the theorem we have

Corollary 1.2. *Let u be a homogeneous order α, $0 < \alpha < 1$ solution of a uniformly elliptic Hessian equation in a punctured ball $B^o \subset \mathbf{R}^n$. Then $u = c|x|^\alpha$.*

The question on the minimal dimension n for which there exist nontrivial homogeneous order 2 solutions of (1) remains open. We notice that from the result of Alexsandrov [A] it follows that any homogeneous order 2 solution of the equation (1) in $\mathbf{R}^3$ with a real analytic F should be a quadratic polynomial. For a smooth and less regular F similar results in the dimension 3 can be found in [HNY].

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